

**TIME SERIES PROPERTIES OF THE CLASS
OF FIRST ORDER AUTOREGRESSIVE
PROCESSES WITH GENERALIZED
MOVING AVERAGE ERRORS**

**THULASYAMMAL RAMIAH PILLAI¹, MAHENDRAN SHITAN^{1, 2}
and SHELTON PEIRIS³**

¹Applied and Computational Statistics Laboratory
Institute for Mathematical Research
Universiti Putra
Malaysia

²Department of Mathematics
Faculty of Science
Universiti Putra
Malaysia
e-mail: mahen698@gmail.com

³School of Mathematics and Statistics
The University of Sydney
New South Wales
Australia

Abstract

A new class of time series models known as Generalized Autoregressive of order one with first order moving average errors has been introduced in order to reveal some features of certain time series data. The variance and autocovariance of the

2000 Mathematics Subject Classification: 62M10.

Keywords and phrases: autoregression, moving average, errors, autocorrelations, variance, autocovariance, spectral density, estimation, time series, fractional differencing, long memory.

Received August 28, 2009

process is derived in order to study the behaviour of the process. It is shown that these new results reduce to the standard ARMA results and therefore, applicable in many special cases. Numerical values of the ACF are obtained and compared with ARMA results.

1. Introduction

It is known that the modelling of time series with changing frequency components is important in many applications, especially in financial data. Although ARMA type models could be used in practice, there is no systematic approach or a suitable class of time series models available in literature to accommodate, analyze, and forecast of such time series. However, in recent years much attention has been focused on the study of fractional differencing and long memory time series to accommodate the low frequency behaviour of time series. In order to investigate all of such properties, (i.e., slowly decaying autocorrelations and/or changing frequencies of a time series) a new class of generalized autoregressive model of order one (GAR(1)) has been introduced by Peiris [2]. This is a natural extension of the standard AR(1) model by adding an additional index parameter $\delta(> 0)$ and is defined by,

$$(1 - \alpha B)^\delta X_t = Z_t, \quad |\alpha| < 1, \quad (1)$$

where $\{Z_t\} \sim WN(0, \sigma^2)$.

The autocovariance function of GAR(1) model in (1) is given by,

$$\gamma_k = \frac{\sigma^2 \alpha^k \Gamma(k + \delta) F(\delta, k + \delta; k + 1; \alpha^2)}{\Gamma(\delta) \Gamma(k + 1)}, \quad k \geq 0, \quad (2)$$

where F is the hypergeometric function defined as,

$$F(a, b; c; z) = 1 + \frac{ab}{1!c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)} z^3 + \dots,$$

or equivalently

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(j+1)\Gamma(c+j)} z^j. \quad (3)$$

It has been shown by Peiris [2] that the model in (1) could be used to model long memory or nearly long memory time series by suitably choosing the parameters α and δ . Shitan and Peiris [7] have studied the estimation problem that arise in class (1) using both the maximum likelihood (MLE) and Whittle procedures. Further, Peiris et al. [3] have introduced the class of generalized moving average of order 1 (GMA(1)), which is given by,

$$X_t = (1 - \beta B)^\delta Z_t. \quad (4)$$

The corresponding autocovariance function of the GMA(1) model is

$$\gamma_k = \frac{\sigma^2 \beta^k \Gamma(k - \delta) F(-\delta, k - \delta; k + 1; \beta^2)}{\Gamma(-\delta) \Gamma(k + 1)}, \quad k \geq 0. \quad (5)$$

It has been shown in Peiris [2] and Peiris et al. [3] that the additional parameter δ plays an important role in modelling and forecasting. We therefore, refer the above classes in (1) and (4) as GAR(1; δ) and GMA(1; δ) for later reference. Peiris et al. [4] and Peiris and Thavaneswaran [5] have shown that the classes given in (1) and (4) could be used to model many time series in practice, especially in finance.

A natural generalization of the standard ARMA (1, 1) using the same approach is called *Generalized Autoregressive Moving Average model* (GARMA). This is denoted by GARMA (1, 1; δ_1, δ_2) and is defined by,

$$(1 - \alpha B)^{\delta_1} X_t = (1 - \beta B)^{\delta_2} Z_t, \quad (6)$$

where $-1 < \alpha, \beta < 1$, $\delta_1 > 0$, and $\delta_2 > 0$.

It is clear that (1) and (4) are special cases of (6). However, this paper considers a special case of (6) with $\delta_1 = 1$, (i.e., GARMA (1, 1; 1, δ)) which is given as,

$$(1 - \alpha B) X_t = (1 - \beta B)^\delta Z_t. \quad (7)$$

It is easy to show that (see, for example, Peiris [2]) the process in (7) has a valid second order stationary solution such that,

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j},$$

where $\psi(B) = (1 - \alpha B)^{-1}(1 - \beta B)^\delta$. Further, the corresponding spectral density function of the process $\{X_t\}$ is,

$$f(\omega) = \frac{\sigma^2(1 - 2\beta \cos \omega + \beta^2)^\delta}{2\pi(1 - 2\alpha \cos \omega + \alpha^2)}; \quad -\pi \leq \omega \leq \pi. \quad (8)$$

See Brockwell and Davis [1] or Priestly [6] for details. It is clear that GARMA(1, 1; 1, δ) reduces to the standard ARMA(1, 1), when $\delta = 1$.

In Section 2, we provide the autocovariance function at lag h , $h \geq 0$. We will also show that these expressions will reduce to the autocovariance function of the standard ARMA.

2. The Theoretical Autocovariance Function of the GARMA(1, 1; 1, δ) Process

Let $\gamma_h = \text{Cov}(X_t, X_{t-h})$ be the autocovariance function at lag h . We first evaluate the variance γ_0 of the GARMA(1, 1; 1, δ) process, which is given in the following proposition.

Proposition 2.1. *For the model defined as in Equation (7), the variance γ_0 is given as,*

$$\begin{aligned} \gamma_0 = & \frac{\sigma^2}{1 - \alpha^2} \left[\sum_{j=1}^{\infty} \binom{\delta}{j} (-\alpha\beta)^j F(-\delta, j - \delta; j + 1; \beta^2) \right. \\ & \left. + \sum_{j=0}^{\infty} \binom{\delta}{j} (-\alpha\beta)^j F(-\delta, j - \delta; j + 1; \beta^2) \right]. \quad (9) \end{aligned}$$

Proof. The proof is established by computing $\gamma_0 = \sigma^2 \sum_{k=0}^{\infty} \psi_k^2$.

Note that the GARMA(1, 1; 1, δ) model can be compactly written as $X_t = \psi(B)Z_t$, where $\psi(B)$ is given by,

$$\psi(B) = \frac{(1 - \beta B)^\delta}{(1 - \alpha B)}.$$

Furthermore, $\psi_0 = 1$ and $\psi_k = \alpha^k + \sum_{j=1}^k \tau_j \beta^j \alpha^{k-j}$ for $k \geq 1$, where

$$\tau_j = \frac{\Gamma(j - \delta)}{\Gamma(j + 1)\Gamma(-\delta)}.$$

Now,

$$\begin{aligned} \sum_{k=1}^{\infty} \psi_k^2 &= \sum_{k=1}^{\infty} \left(\alpha^k + \sum_{j=1}^k \tau_j \beta^j \alpha^{k-j} \right)^2 \\ &= \sum_{k=1}^{\infty} \left[\alpha^{2k} + 2\alpha^k \sum_{j=1}^k \tau_j \beta^j \alpha^{k-j} + \left(\sum_{j=1}^k \tau_j \beta^j \alpha^{k-j} \right)^2 \right] \\ &= \sum_{k=1}^{\infty} \alpha^{2k} + 2 \sum_{k=1}^{\infty} \alpha^k \sum_{j=1}^k \tau_j \beta^j \alpha^{k-j} + \sum_{k=1}^{\infty} \left(\sum_{j=1}^k \tau_j \beta^j \alpha^{k-j} \right)^2 \\ &= \frac{\alpha^2}{1 - \alpha^2} + \frac{2}{1 - \alpha^2} \sum_{j=1}^{\infty} \tau_j (\alpha\beta)^j + \frac{1}{1 - \alpha^2} [F(-\delta, -\delta; 1; \beta^2) - 1] \\ &\quad + \frac{2}{1 - \alpha^2} \sum_{j=1}^{\infty} (\alpha\beta)^j \tau_j [F(-\delta, j - \delta; j + 1; \beta^2) - 1]. \end{aligned} \quad (10)$$

From (10), we note that

$$\begin{aligned} (1 - \alpha^2) \sum_{k=1}^{\infty} \psi_k^2 &= \alpha^2 + 2 \left[\sum_{j=1}^{\infty} \frac{(\alpha\beta)^j \Gamma(j - \delta)}{\Gamma(-\delta)\Gamma(j + 1)} [F(-\delta, j - \delta; j + 1; \beta^2) - 1] \right] \\ &\quad + 2 \sum_{j=1}^{\infty} (\alpha\beta)^j \tau_j + F(-\delta, -\delta; 1; \beta^2) - 1. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{k=1}^{\infty} \psi_k^2 &= \frac{\alpha^2 + 2 \left[\sum_{j=1}^{\infty} \frac{(\alpha\beta)^j \Gamma(j-\delta)}{\Gamma(-\delta)\Gamma(j+1)} F(-\delta, j-\delta; j+1; \beta^2) \right] + F(-\delta, -\delta; 1; \beta^2) - 1}{1 - \alpha^2} \\ &= \frac{2 \left[\sum_{j=1}^{\infty} \frac{(\alpha\beta)^j \Gamma(j-\delta)}{\Gamma(-\delta)\Gamma(j+1)} F(-\delta, j-\delta; j+1; \beta^2) \right] + F(-\delta, -\delta; 1; \beta^2)}{1 - \alpha^2} - 1. \end{aligned}$$

Therefore, we obtain,

$$\sum_{k=0}^{\infty} \psi_k^2 = \frac{2 \left[\sum_{j=1}^{\infty} \frac{(\alpha\beta)^j \Gamma(j-\delta)}{\Gamma(-\delta)\Gamma(j+1)} F(-\delta, j-\delta; j+1; \beta^2) \right] + F(-\delta, -\delta; 1; \beta^2)}{1 - \alpha^2},$$

and from which the result follows

$$\sum_{k=0}^{\infty} \psi_k^2 = \frac{\sum_{j=1}^{\infty} \frac{(\alpha\beta)^j \Gamma(j-\delta)}{\Gamma(-\delta)\Gamma(j+1)} F(-\delta, j-\delta; j+1; \beta^2) + \sum_{j=0}^{\infty} \frac{(\alpha\beta)^j \Gamma(j-\delta)}{\Gamma(-\delta)\Gamma(j+1)} F(-\delta, j-\delta; j+1; \beta^2)}{1 - \alpha^2}.$$

Hence,

$$\begin{aligned} \gamma_0 &= \frac{\sigma^2}{1 - \alpha^2} \left[\sum_{j=1}^{\infty} \binom{\delta}{j} (-\alpha\beta)^j F(-\delta, j-\delta; j+1; \beta^2) \right. \\ &\quad \left. + \sum_{j=0}^{\infty} \binom{\delta}{j} (-\alpha\beta)^j F(-\delta, j-\delta; j+1; \beta^2) \right]. \quad \blacksquare \end{aligned}$$

Remark. Note that as a consequence of Proposition 2.1, we have obtained a formula for following integral.

$$\begin{aligned} \int_0^{\pi} \frac{(1 - 2\beta \cos \omega + \beta^2)^{\delta}}{(1 - 2\alpha \cos \omega + \alpha^2)} d\omega &= \frac{\pi}{1 - \alpha^2} \left[\sum_{j=1}^{\infty} \binom{\delta}{j} (-\alpha\beta)^j F(-\delta, j-\delta; j+1; \beta^2) \right. \\ &\quad \left. + \sum_{j=0}^{\infty} \binom{\delta}{j} (-\alpha\beta)^j F(-\delta, j-\delta; j+1; \beta^2) \right]. \end{aligned}$$

Proposition 2.2. *For the model defined in Equation (7), the autocovariance at lag h is given by,*

$$\begin{aligned} \gamma_h &= \frac{\sigma^2}{1 - \alpha^2} \left[\beta^h \sum_{j=1}^{\infty} \binom{\delta}{h+j} (-\alpha\beta)^j F(-\delta, h+j-\delta; h+j+1; \beta^2) \right. \\ &\quad + \alpha^h \sum_{j=0}^{\infty} \binom{\delta}{j} (-\alpha\beta)^j F(-\delta, j-\delta; j+1; \beta^2) \\ &\quad \left. + \sum_{j=1}^h \binom{\delta}{j} \alpha^{h-j} (-\beta)^j F(-\delta, j-\delta; j+1; \beta^2) \right], \quad h \geq 1. \end{aligned} \quad (11)$$

Proof. The proof is established by computing $\gamma_h = \sigma^2 \sum_{k=0}^{\infty} \psi_k \psi_{k+h}$.

$$\begin{aligned} \sum_{k=1}^{\infty} \psi_k \psi_{k+h} &= \sum_{k=1}^{\infty} \left(\alpha^k + \sum_{j=1}^k \tau_j \beta^j \alpha^{k-j} \right) \left(\alpha^{k+h} + \sum_{j=1}^{k+h} \tau_j \beta^j \alpha^{k+h-j} \right) \\ &= \sum_{k=1}^{\infty} \left[\alpha^k \alpha^{k+h} + \alpha^k \sum_{j=1}^{k+h} \tau_j \beta^j \alpha^{k+h-j} + \alpha^{k+h} \sum_{j=1}^k \tau_j \beta^j \alpha^{k-j} \right. \\ &\quad \left. + \sum_{j=1}^k \tau_j \beta^j \alpha^{k-j} \sum_{j=1}^{k+h} \tau_j \beta^j \alpha^{k+h-j} \right] \\ &= \sum_{k=1}^{\infty} \alpha^k \alpha^{k+h} + \sum_{k=1}^{\infty} \alpha^k \sum_{j=1}^{k+h} \tau_j \beta^j \alpha^{k+h-j} + \sum_{k=1}^{\infty} \alpha^{k+h} \sum_{j=1}^k \tau_j \beta^j \alpha^{k-j} \\ &\quad + \sum_{k=1}^{\infty} \left(\sum_{j=1}^k \tau_j \beta^j \alpha^{k-j} \sum_{j=1}^{k+h} \tau_j \beta^j \alpha^{k+h-j} \right) \\ &= \alpha^h \left(\frac{\alpha^2}{1 - \alpha^2} \right) + \frac{1}{1 - \alpha^2} \left(\sum_{j=1}^{\infty} \tau_{j+h} \beta^{j+h} \alpha^j + \sum_{j=1}^h \tau_j \beta^j \alpha^{h+2-j} \right) \\ &\quad + \frac{\alpha^h}{1 - \alpha^2} \sum_{j=1}^{\infty} \tau_j (\alpha\beta)^j \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{1-\alpha^2} \sum_{j=1}^h \alpha^{h-j} \beta^j \tau_j [F(-\delta, j-\delta; j+1; \beta^2) - 1] \\
& + \frac{\alpha^h}{1-\alpha^2} \sum_{j=0}^{\infty} (\alpha\beta)^j \tau_j [F(-\delta, j-\delta; j+1; \beta^2) - 1] \\
& + \frac{\beta^h}{1-\alpha^2} \sum_{j=1}^{\infty} (\alpha\beta)^j \tau_{h+j} [F(-\delta, h+j-\delta; h+j+1; \beta^2) - 1] \\
= & \frac{1}{1-\alpha^2} \left[\alpha^{h+2} + \sum_{j=1}^h \tau_j \beta^j \alpha^{h+2-j} + \sum_{j=1}^{\infty} \tau_{h+j} \beta^{h+j} \alpha^j \right. \\
& + \alpha^h \sum_{j=1}^{\infty} \tau_j (\alpha\beta)^j + \sum_{j=1}^h \alpha^{h-j} \beta^j \tau_j F(-\delta, j-\delta; j+1; \beta^2) \\
& - \sum_{j=1}^h \alpha^{h-j} \beta^j \tau_j + \alpha^h \sum_{j=0}^{\infty} (\alpha\beta)^j \tau_j F(-\delta, j-\delta; j+1; \beta^2) \\
& - \alpha^h \sum_{j=0}^{\infty} (\alpha\beta)^j \tau_j + \beta^h \sum_{j=1}^{\infty} (\alpha\beta)^j \tau_{h+j} F(-\delta, h+j-\delta; h+j+1; \beta^2) \\
& \left. - \beta^h \sum_{j=1}^{\infty} (\alpha\beta)^j \tau_{h+j} \right] \\
= & \frac{1}{1-\alpha^2} \left[\alpha^{h+2} - \alpha^h + \sum_{j=1}^h \tau_j \beta^j \alpha^{h+2-j} + \sum_{j=1}^{\infty} \tau_{h+j} \beta^{h+j} \alpha^j \right. \\
& + \alpha^h \sum_{j=0}^{\infty} \tau_j (\alpha\beta)^j + \sum_{j=1}^h \alpha^{h-j} \beta^j \tau_j F(-\delta, j-\delta; j+1; \beta^2) \\
& \left. - \sum_{j=1}^h \alpha^{h-j} \beta^j \tau_j + \alpha^h \sum_{j=0}^{\infty} (\alpha\beta)^j \tau_j F(-\delta, j-\delta; j+1; \beta^2) \right]
\end{aligned}$$

$$\begin{aligned}
& -\alpha^h \sum_{j=0}^{\infty} (\alpha\beta)^j \tau_j + \beta^h \sum_{j=1}^{\infty} (\alpha\beta)^j \tau_{h+j} F(-\delta, h+j-\delta; h+j+1; \beta^2) \\
& - \sum_{j=1}^{\infty} \alpha^j \beta^{h+j} \tau_{h+j}] \\
= & \frac{1}{1-\alpha^2} [\alpha^h (\alpha^2 - 1) + (\alpha^2 - 1) \sum_{j=1}^h \tau_j \beta^j \alpha^{h-j} \\
& + \sum_{j=1}^h \alpha^{h-j} \beta^j \tau_j F(-\delta, j-\delta; j+1; \beta^2) \\
& + \alpha^h \sum_{j=0}^{\infty} (\alpha\beta)^j \tau_j F(-\delta, j-\delta; j+1; \beta^2) \\
& + \beta^h \sum_{j=1}^{\infty} (\alpha\beta)^j \tau_{h+j} F(-\delta, h+j-\delta; h+j+1; \beta^2)] \\
= & \frac{1}{1-\alpha^2} [(\alpha^2 - 1)(\alpha^h + \sum_{j=1}^h \tau_j \beta^j \alpha^{h-j}) \\
& + \sum_{j=1}^h \alpha^{h-j} \beta^j \tau_j F(-\delta, j-\delta; j+1; \beta^2) \\
& + \alpha^h \sum_{j=0}^{\infty} (\alpha\beta)^j \tau_j F(-\delta, j-\delta; j+1; \beta^2) \\
& + \beta^h \sum_{j=1}^{\infty} (\alpha\beta)^j \tau_{h+j} F(-\delta, h+j-\delta; h+j+1; \beta^2)].
\end{aligned}$$

Finally, it can be reduced to

$$\sum_{k=1}^{\infty} \psi_k \psi_{k+h} = \frac{1}{1-\alpha^2} [(\alpha^2 - 1)(\psi_h)]$$

$$\begin{aligned}
& + \sum_{j=1}^h \alpha^{h-j} \beta^j {}_j F(-\delta, j-\delta; j+1; \beta^2) \\
& + \alpha^h \sum_{j=0}^{\infty} (\alpha\beta)^j {}_j F(-\delta, j-\delta; j+1; \beta^2) \\
& + \beta^h \sum_{j=1}^{\infty} (\alpha\beta)^j {}_{h+j} F(-\delta, h+j-\delta; h+j+1; \beta^2)].
\end{aligned}$$

Hence, we obtain,

$$\begin{aligned}
\sum_{k=0}^{\infty} \psi_k \psi_{k+h} & = \frac{1}{1-\alpha^2} \left[\sum_{j=1}^h \alpha^{h-j} \beta^j {}_j F(-\delta, j-\delta; j+1; \beta^2) \right. \\
& + \alpha^h \sum_{j=0}^{\infty} (\alpha\beta)^j {}_j F(-\delta, j-\delta; j+1; \beta^2) \\
& \left. + \beta^h \sum_{j=1}^{\infty} (\alpha\beta)^j {}_{h+j} F(-\delta, h+j-\delta; h+j+1; \beta^2) \right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\gamma_h & = \frac{\sigma^2}{1-\alpha^2} \left[\beta^h \sum_{j=1}^{\infty} \binom{\delta}{j+h} (-\alpha\beta)^j {}_j F(-\delta, h+j-\delta; h+j+1; \beta^2) \right. \\
& + \alpha^h \sum_{j=0}^{\infty} \binom{\delta}{j} (-\alpha\beta)^j {}_j F(-\delta, j-\delta; j+1; \beta^2) \\
& \left. + \sum_{j=1}^h \binom{\delta}{j} \alpha^{h-j} (-\beta)^j {}_j F(-\delta, j-\delta; j+1; \beta^2) \right], \quad h \geq 1,
\end{aligned}$$

which completes the proof.

3. Special Cases

In this section, we shall show that the variance and autocovariance function given by Proposition 2.1 (Equation 9) and Proposition 2.2 (Equation 11) correctly reduces to the variance and autocovariance function of the special cases considered in this section.

3.1. White noise model

In the special case, when $\alpha = \beta = 0$ and $\delta = 1$, the GARMA(1, 1; 1, δ) model reduces to white noise. By substituting $\alpha = \beta = 0$ and $\delta = 1$ into (9), we obtain

$$\begin{aligned}\gamma_0 &= \frac{\sigma^2}{1 - (0)^2} \left[\sum_{j=1}^{\infty} \binom{1}{j} (- (0)(0))^j F(-1, j-1; j+1; (0)^2) \right. \\ &\quad \left. + \sum_{j=0}^{\infty} \binom{1}{j} (- (0)(0))^j F(-1, j-1; j+1; (0)^2) \right] \\ &= \sigma^2 \binom{1}{0} (- (0)(0))^0 F(-1, 0-1; 0+1; (0)^2) \\ &= \sigma^2.\end{aligned}$$

Again by substituting $\alpha = \beta = 0$ and $\delta = 1$ into (11), we obtain

$$\begin{aligned}\gamma_h &= \frac{\sigma^2}{1 - (0)^2} \left[(0)^h \sum_{j=1}^{\infty} \binom{1}{h+j} (- (0)(0))^j F(-1, h+j-1; h+j+1; (0)^2) \right. \\ &\quad \left. + (0)^h \sum_{j=0}^{\infty} \binom{1}{j} (- (0)(0))^j F(-1, j-1; j+1; (0)^2) \right. \\ &\quad \left. + \sum_{j=1}^{\infty} \binom{1}{j} (0)^{h-j} (-0)^j F(-1, j-1; j+1; (0)^2) \right] \\ &= 0.\end{aligned}$$

3.2. Moving average (MA(1)) model

In addition, GARMA(1, 1; 1, δ) also covers the MA(1) family when $\alpha = 0$ and $\delta = 1$. By substituting $\alpha = 0$ and $\delta = 1$ into (9), we get,

$$\begin{aligned}\gamma_0 &= \frac{\sigma^2}{1 - (0)^2} \left[\sum_{j=1}^{\infty} \binom{1}{j} (- (0)\beta)^j F(-1, j-1; j+1; \beta^2) \right. \\ &\quad \left. + \sum_{j=0}^{\infty} \binom{1}{j} (- (0)\beta)^j F(-1, j-1; j+1; \beta^2) \right] \\ &= \sigma^2 \binom{1}{0} (- (0)\beta)^0 F(-1, 0-1; 0+1; \beta^2) \\ &= \sigma^2 (1 + \beta^2).\end{aligned}$$

Again by substituting $\alpha = 0$ and $\delta = 1$ into (11), we get,

$$\begin{aligned}\gamma_h &= \frac{\sigma^2}{1 - (0)^2} \left[\beta^h \sum_{j=1}^{\infty} \binom{1}{h+j} (- (0)\beta)^j F(-1, h+j-1; h+j+1; \beta^2) \right. \\ &\quad \left. + (0)^h \sum_{j=0}^{\infty} \binom{1}{j} (- (0)\beta)^j F(-1, j-1; j+1; \beta^2) \right. \\ &\quad \left. + \sum_{j=1}^h \binom{1}{j} (0)^{h-j} (-\beta)^j F(-1, j-1; j+1; \beta^2) \right] \\ &= \frac{\sigma^2}{1 - (0)^2} \left[\sum_{j=1}^h \binom{1}{j} (0)^{h-j} (-\beta)^j F(-1, j-1; j+1; \beta^2) \right] \\ &= \sigma^2 \left[\binom{1}{h} (-\beta)^h F(-1, h-1; h+1; \beta^2) \right].\end{aligned}\tag{12}$$

Equation (12) reduces to, $\gamma_h = -\beta\sigma^2$ when $h = 1$ and $\gamma_h = 0$ when $h \geq 2$.

3.3. ARMA(1,1) model

In the special case, when $\delta = 1$, then the GARMA(1, 1; 1, δ) model reduces to ARMA(1,1). By substituting $\delta = 1$ into (9), we obtain,

$$\begin{aligned}
\gamma_0 &= \frac{\sigma^2}{1 - \alpha^2} \left[\sum_{j=1}^{\infty} \binom{1}{j} (-\alpha\beta)^j F(-1, j-1; j+1; \beta^2) \right. \\
&\quad \left. + \sum_{j=0}^{\infty} \binom{1}{j} (-\alpha\beta)^j F(-1, j-1; j+1; \beta^2) \right] \\
&= \frac{\sigma^2}{1 - \alpha^2} \left[\binom{1}{1} (-\alpha\beta)^1 F(-1, 1-1; 1+1; \beta^2) \right. \\
&\quad \left. + \binom{1}{0} (-\alpha\beta)^0 F(-1, 0-1; 0+1; \beta^2) \right. \\
&\quad \left. + \binom{1}{1} (-\alpha\beta)^1 F(-1, 1-1; 1+1; \beta^2) \right] \\
&= \frac{\sigma^2}{1 - \alpha^2} [-\alpha\beta + 1 + \beta^2 - \alpha\beta] \\
&= \frac{\sigma^2}{1 - \alpha^2} [1 + \beta^2 - 2\alpha\beta] \\
&= \frac{\sigma^2}{1 - \alpha^2} [1 - \alpha^2 + \alpha^2 + \beta^2 - 2\alpha\beta] \\
&= \sigma^2 \left[1 + \frac{(\alpha - \beta)^2}{1 - \alpha^2} \right].
\end{aligned}$$

Again by substituting $\delta = 1$ into (11), we get,

$$\begin{aligned}
\gamma_h &= \frac{\sigma^2}{1 - \alpha^2} \left[\beta^h \sum_{j=1}^{\infty} \binom{1}{h+j} (-\alpha\beta)^j F(-1, h+j-1; h+j+1; \beta^2) \right. \\
&\quad \left. + \alpha^h \sum_{j=0}^{\infty} \binom{1}{j} (-\alpha\beta)^j F(-1, j-1; j+1; \beta^2) \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^h \binom{1}{j} \alpha^{h-j} (-\beta)^j F(-1, j-1; j+1; \beta^2)], \\
\gamma_h & = \frac{\sigma^2}{1-\alpha^2} \left[\alpha^h \sum_{j=0}^{\infty} \binom{1}{j} (-\alpha\beta)^j F(-1, j-1; j+1; \beta^2) \right. \\
& \quad \left. + \sum_{j=1}^h \binom{1}{j} \alpha^{h-j} (-\beta)^j F(-1, j-1; j+1; \beta^2) \right], \\
\gamma_h & = \frac{\sigma^2}{1-\alpha^2} \left[\alpha^h \left[\binom{1}{0} (-\alpha\beta)^0 F(-1, 0-1; 0+1; \beta^2) \right. \right. \\
& \quad \left. \left. + \binom{1}{1} (-\alpha\beta)^1 F(-1, 1-1; 1+1; \beta^2) \right] \right. \\
& \quad \left. + \binom{1}{1} \alpha^{h-1} (-\beta)^1 F(-1, 1-1; 1+1; \beta^2) \right], \\
\gamma_h & = \frac{\sigma^2}{1-\alpha^2} \left[\alpha^h [F(-1, -1; +1; \beta^2) + (-\alpha\beta)^1 F(-1, 0; 2; \beta^2)] \right. \\
& \quad \left. + \alpha^{h-1} (-\beta)^1 F(-1, 0; 2; \beta^2) \right] \\
& = \frac{\sigma^2}{1-\alpha^2} \left[\alpha^h [(1+\beta^2) - \alpha\beta] - \alpha^{h-1}\beta \right] \\
& = \frac{\sigma^2 \alpha^{h-1}}{1-\alpha^2} [\alpha + \alpha\beta^2 - \alpha^2\beta - \beta] \\
& = \sigma^2 \alpha^{h-1} \left[\alpha - \beta + \frac{\alpha(\alpha - \beta)^2}{1-\alpha^2} \right] \\
& = \alpha^{h-1} \gamma_1.
\end{aligned}$$

3.4. Generalized moving average (GMA(1)) model

The GARMA(1, 1; 1, δ) also reduces to the GMA(1) model when $\alpha = 0$. By substituting $\alpha = 0$ into (9), we obtain,

$$\begin{aligned}
\gamma_0 &= \frac{\sigma^2}{1 - 0^2} \left[\sum_{j=1}^{\infty} \binom{\delta}{j} (- (0)\beta)^j F(-\delta, j - \delta; j + 1; \beta^2) \right. \\
&\quad \left. + \sum_{j=0}^{\infty} \binom{\delta}{j} (- (0)\beta)^j F(-\delta, j - \delta; j + 1; \beta^2) \right] \\
&= \sigma^2 \binom{\delta}{0} (- (0)\beta)^0 F(-\delta, 0 - \delta; 0 + 1; \beta^2) \\
&= \sigma^2 F(-\delta, -\delta; 1; \beta^2).
\end{aligned}$$

Again by substituting $\alpha = 0$ into (11), we obtain,

$$\begin{aligned}
\gamma_h &= \frac{\sigma^2}{1 - (0)^2} \left[\beta^h \sum_{j=1}^{\infty} \binom{\delta}{h+j} (- (0)\beta)^j F(-\delta, h + j - \delta; h + j + 1; \beta^2) \right. \\
&\quad \left. + (0)^h \sum_{j=0}^{\infty} \binom{\delta}{j} (- (0)\beta)^j F(-\delta, j - \delta; j + 1; \beta^2) \right. \\
&\quad \left. + \sum_{j=1}^h \binom{\delta}{j} (0)^{h-j} (-\beta)^j F(-\delta, j - \delta; j + 1; \beta^2) \right] \\
&= \sigma^2 \left[\sum_{j=1}^h \binom{\delta}{j} (0)^{h-j} (-\beta)^j F(-\delta, j - \delta; j + 1; \beta^2) \right] \\
&= \sigma^2 \left[\binom{\delta}{h} (-\beta)^h F(-\delta, h - \delta; h + 1; \beta^2) \right] \\
&= \frac{\sigma^2 \beta^h \Gamma(h - \delta) F(-\delta, h - \delta; h + 1; \beta^2)}{\Gamma(-\delta) \Gamma(h + 1)}, \quad h \geq 1.
\end{aligned}$$

The above expression agrees with the autocovariance function as given in Peiris et al. [3].

4. Numerical Results

In this section, we tabulate some numerical results.

In Tables 1 and 2, the theoretical variance computed from Equation (9) are given.

Table 1. The values of $\gamma_0(\alpha = 0.9, \beta = 0.5, \sigma^2 = 1.0)$

h	$\delta = 0.2$	$\delta = 0.4$	$\delta = 0.6$	$\delta = 0.8$	$\delta = 1.0$	$\delta = 1.4$	$\delta = 1.8$
0	4.1603	3.3200	2.6823	2.2013	1.8421	1.3899	1.1828

Table 2. The values of $\gamma_0(\alpha = 0.95, \beta = 0.8, \sigma^2 = 1.0)$

h	$\delta = 0.2$	$\delta = 0.4$	$\delta = 0.6$	$\delta = 0.8$	$\delta = 1.0$	$\delta = 1.4$	$\delta = 1.8$
0	5.8816	3.5139	2.2397	1.5668	1.2308	1.0677	1.2489

The theoretical autocorrelation function is given by, $\rho_h = \gamma_h / \gamma_0$ and its values are tabulated in Tables 3 and 4.

Table 3. The values of $\rho_h(\alpha = 0.9, \beta = 0.5, \sigma^2 = 1.0)$

h	$\delta = 0.2$	$\delta = 0.4$	$\delta = 0.6$	$\delta = 0.8$	$\delta = 1.0$	$\delta = 1.4$	$\delta = 1.8$
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.8706	0.8314	0.7796	0.7127	0.6286	0.4080	0.1402
2	0.7771	0.7368	0.6883	0.6311	0.5657	0.4189	0.2785
3	0.6973	0.6599	0.6162	0.5658	0.5091	0.3823	0.2557
4	0.6268	0.5928	0.5536	0.5087	0.4582	0.3452	0.2310
5	0.5639	0.5332	0.4979	0.4575	0.4124	0.3110	0.2080
6	0.5073	0.4797	0.4479	0.4118	0.3712	0.2799	0.1873
7	0.4566	0.4317	0.4031	0.3706	0.3341	0.2520	0.1686
8	0.4109	0.3885	0.3627	0.3335	0.3006	0.2268	0.1517
9	0.3698	0.3496	0.3265	0.3001	0.2706	0.2041	0.1365
10	0.3328	0.3146	0.2938	0.2701	0.2435	0.1837	0.1228
11	0.2995	0.2832	0.2644	0.2431	0.2192	0.1653	0.1106
12	0.2696	0.2548	0.2380	0.2188	0.1973	0.1488	0.0995

13	0.2426	0.2294	0.2142	0.1969	0.1775	0.1339	0.0896
14	0.2183	0.2064	0.1928	0.1772	0.1598	0.1205	0.0807
15	0.1965	0.1858	0.1735	0.1595	0.1438	0.1085	0.0725
16	0.1769	0.1672	0.1561	0.1436	0.1294	0.0976	0.0653
17	0.1592	0.1505	0.1405	0.1292	0.1165	0.0878	0.0588
18	0.1433	0.1355	0.1265	0.1163	0.1048	0.0791	0.0529
19	0.1289	0.1219	0.1138	0.1047	0.0943	0.0712	0.0476
20	0.1160	0.1097	0.1024	0.0942	0.0849	0.0640	0.0429

Table 4. The values of $\rho_h(\alpha = 0.95, \beta = 0.8, \sigma^2 = 1.0)$

h	$\delta = 0.2$	$\delta = 0.4$	$\delta = 0.6$	$\delta = 0.8$	$\delta = 1.0$	$\delta = 1.4$	$\delta = 1.8$
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.9092	0.8351	0.7105	0.5268	0.3000	-0.1263	-0.3837
2	0.8476	0.7598	0.6299	0.4623	0.2850	0.0439	-0.0052
3	0.7966	0.7059	0.5799	0.4261	0.2708	0.0673	0.0125
4	0.7515	0.6617	0.5414	0.3987	0.2572	0.0721	0.0158
5	0.7106	0.6233	0.5090	0.3756	0.2444	0.0718	0.0163
6	0.6728	0.5887	0.4803	0.3549	0.2321	0.0698	0.0160
7	0.6377	0.5570	0.4542	0.3360	0.2205	0.0672	0.0155
8	0.6047	0.5276	0.4301	0.3185	0.2095	0.0643	0.0149
9	0.5737	0.5001	0.4077	0.3021	0.1990	0.0614	0.0142
10	0.5445	0.4744	0.3867	0.2867	0.1891	0.0585	0.0135
11	0.5169	0.4501	0.3669	0.2721	0.1796	0.0557	0.0129
12	0.4908	0.4273	0.3483	0.2584	0.1706	0.0529	0.0122
13	0.4660	0.4056	0.3306	0.2454	0.1621	0.0503	0.0116
14	0.4426	0.3851	0.3139	0.2330	0.1540	0.0479	0.0111
15	0.4203	0.3657	0.2981	0.2213	0.1463	0.0455	0.0105
16	0.3992	0.3473	0.2831	0.2102	0.1390	0.0432	0.0100
17	0.3792	0.3299	0.2689	0.1997	0.1320	0.0411	0.0095
18	0.3602	0.3133	0.2554	0.1897	0.1254	0.0390	0.0090
19	0.3421	0.2976	0.2426	0.1802	0.1192	0.0371	0.0086
20	0.3250	0.2827	0.2305	0.1712	0.1132	0.0352	0.0081

Plots of the ACF are shown in Figures 1-2. From both the figures, we observe that when $\delta > 1.0$, the ACF lies below that of the standard ARMA model. That is the ACF decays quicker than the standard ARMA model. On the other hand, when $\delta < 1.0$, the ACF lies above that of the standard ARMA model. That is, the ACF decays slower than the standard ARMA model. Hence, this model can be used to model many type of autocorrelation structures.

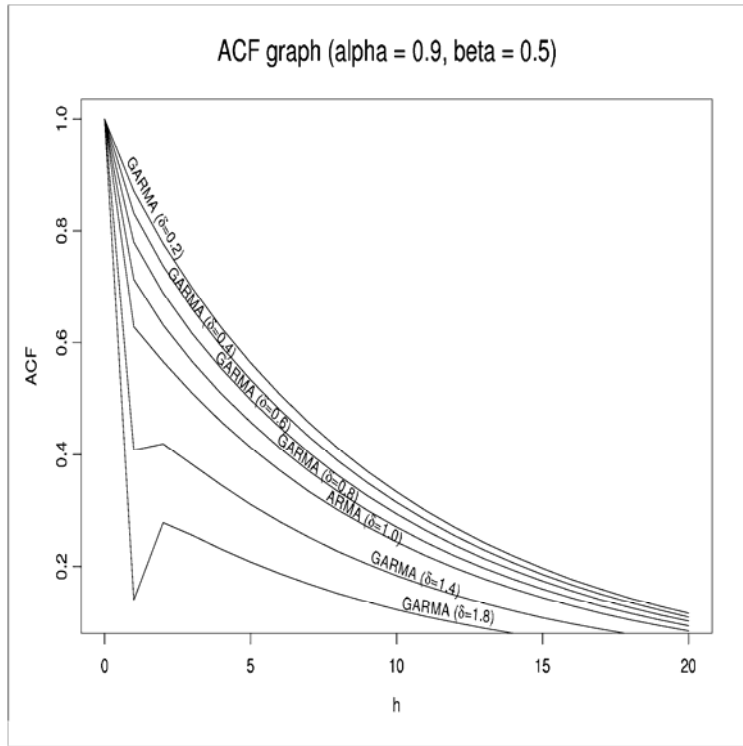


Figure 1. ACF Plot.

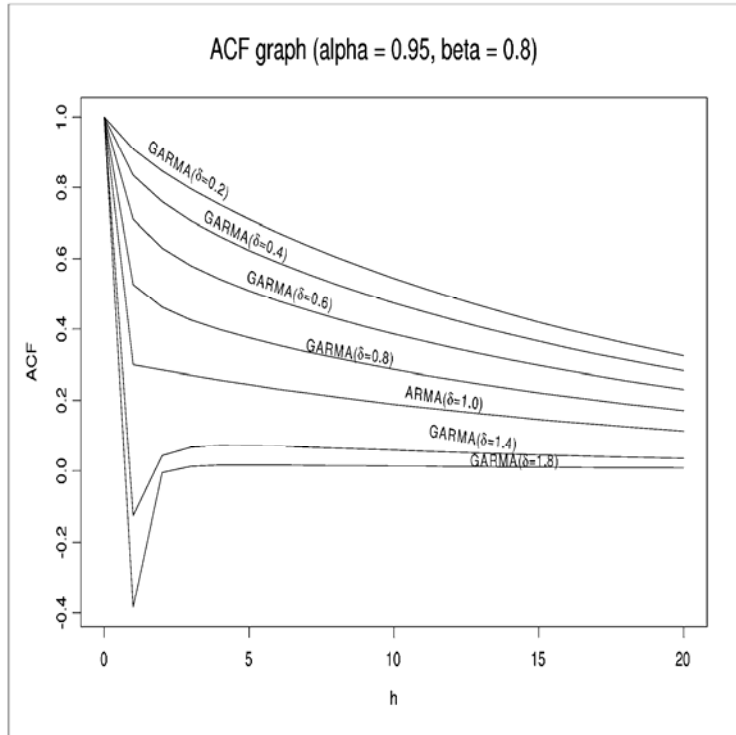


Figure 2. ACF Plot.

In Figure 3, the model spectrum of standard ARMA(1, 1) ($\alpha = 0.9$, $\beta = 0.8$) and GARMA(1, 1; 1, δ) ($\alpha = 0.9$, $\beta = 0.8$, $\delta = 0.8$, and $\alpha = 0.9$, $\beta = 0.8$, $\delta = 1.2$) are plotted. We notice in Figure 3, that the low frequencies dominate. For GARMA model, when δ is greater than one the spectrum lies below the spectrum of standard ARMA, but when δ is smaller than one the spectrum lies above the spectrum of standard ARMA at low frequencies.

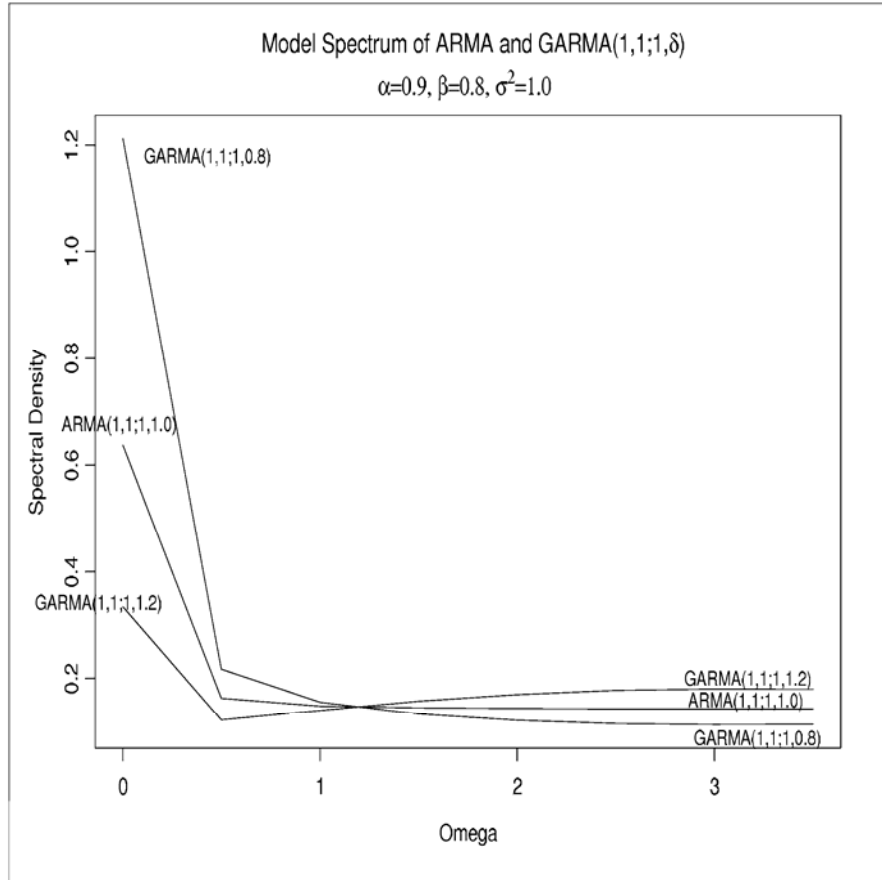


Figure 3. Model spectrum of standard ARMA($\alpha = 0.9$, $\beta = 0.8$, $\sigma^2 = 1.0$) and GARMA($\alpha = 0.9$, $\beta = 0.8$, $\sigma^2 = 1.0$, $\delta = 0.8$, and $\delta = 1.2$).

In Figure 4, the model spectrum of standard ARMA(1, 1) ($\alpha = -0.9$, $\beta = 0.8$) and GARMA(1, 1; 1, δ) ($\alpha = -0.9$, $\beta = 0.8$, $\delta = 0.8$, and $\alpha = 0.9$, $\beta = 0.8$, $\delta = 1.2$) are plotted. We notice in Figure 4, that the high frequencies dominate. For GARMA model, when δ is greater than one the spectrum lies above the spectrum of standard ARMA, but when δ is smaller than one the spectrum lies below the spectrum of standard ARMA at high frequencies.

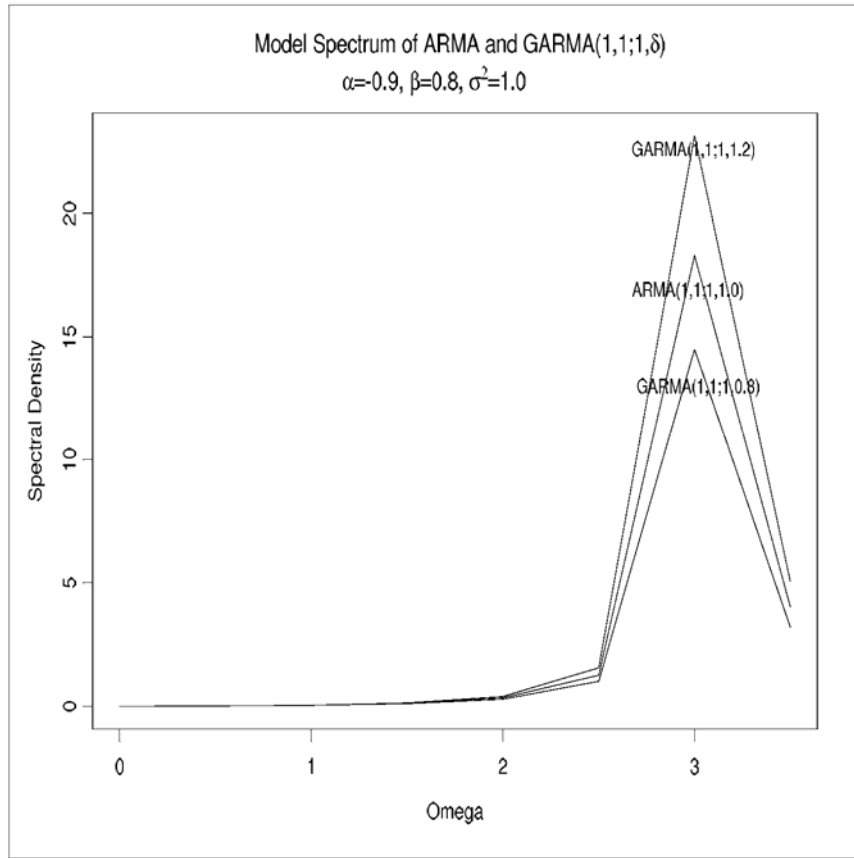


Figure 4. Model spectrum of standard ARMA ($\alpha = -0.9, \beta = 0.8, \sigma^2 = 1.0$) and GARMA ($\alpha = -0.9, \beta = 0.8, \sigma^2 = 1.0, \delta = 0.8$, and $\delta = 1.2$).

5. Conclusion

The objective of this paper was to establish the variance and autocovariance function of the GARMA(1, 1; 1, δ) process. The important results are contained in the propositions. We have also successfully verified that our propositions are correct in the special cases considered in Section 3. These results contribute to the theory of the model considered in this paper and would be useful in modelling certain time series data.

The authors are currently investigating further properties of the model and shall be reported in a future paper.

References

- [1] P. J. Brockwell and R. A. Davis, Introduction to Time Series and Forecasting, 2nd Edition, Springer, New York, (2002).
- [2] M. S. Peiris, Improving the quality of forecasting using generalized AR models: An application to statistical quality control, *Statistical Methods* 5(2) (2003), 156-171.
- [3] M. S. Peiris, D. Allen and A. Thavaneswaran, An introduction to generalized moving average models and applications, *J. Appl. Statist. Sci.* 13(3) (2004), 251-267.
- [4] M. S. Peiris, D. Allen and U. Peiris, Generalized autoregressive models with conditional heteroscedasticity: An application to financial time series modelling, *Proceedings of the 2004 Workshop on Research Methods: Statistics and Finance* (2005), 75-83.
- [5] M. S. Peiris and A. Thavaneswaran, An introduction to volatility models with indicies, *Appl. Math. Lett.* 20 (2007), 177-182.
- [6] M. B. Priestly, *Spectral Analysis and Time Series*, Academic Press, New York, (1981).
- [7] M. Shitan and S. Peiris, Generalised autoregressive (GAR) model: A comparison of maximum likelihood and Whittle estimation procedures using a simulation study, *Commun. Statist. -Simulation and Computation* 37(3) (2008), 560-570.

